Edge-partitions of sparse graphs and their applications to game coloring

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Abstract

In this note, we prove that every graph with maximum average degree less than $\frac{32}{13}$ (resp. $\frac{30}{11}$, $\frac{32}{11}$, $\frac{70}{23}$) admits an edge-partition into a forest and a subgraph of maximum degree 1 (resp. 2, 3, 4). This implies that these graphs have game coloring number at most 5, 6, 7, 8, respectively.

1 Introduction

Let G be a simple graph. The *game coloring number* of G is defined through a two-person graph ordering game. Alice and Bob take turns choosing vertices from the set of unchosen vertices of G. This defines a linear order L of the vertices of G with $x < y$, if and only if, x is chosen before y. The *back degree* of a vertex x with respect to L is the number of its neighbors y in G such that $y < x$. The back degree of L is the maximum back degree of a vertex of G with respect to L. Alice's goal is to minimize the back degree of L and Bob's goal is to maximize it. The *game coloring number* $col_g(G)$ of G is defined to be $k+1$, where k is the minimum integer such that Alice has a strategy for the graph ordering game to ensure that the back degree of L is at most k. Equivalently, k is the maximum integer such that Bob has a strategy for the graph ordering game to ensure that the back degree of L is at least k. This notion was first formally defined in [5] as a tool to find bounds to the game chromatic number [1].

Recently, Zhu [6] proved that the game coloring number of every planar graph is at most 17. This result was improved in the case of planar graphs with large girth, by Borodin *et al.* [2] and He *et al.* [4]. These results are based on some structural properties of planar graphs with large girth:

Theorem 1 (Borodin *et. al.* [2] + He *et. al.* [4]) *Let G be a planar graph with girth at least g.*

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1. *If $g \geq 9$, then G admits an edge-partition into a forest and a matching [2].*
2. *If $g \geq 7$, then G admits an edge-partition into a forest and a graph with maximum degree 2 [4].*
3. *If $g \geq 5$, then G admits an edge-partition into a forest and a graph with maximum degree 4 [4].*

Zhu established this upper bound of the game coloring number:

Lemma 1 (Zhu [5]) *Suppose that the graph G has an edge-partition into two subgraphs G_1 and G_2, then $col_g(G) \leq col_g(G_1) + \Delta(G_2)$.*

Faigle *et al.* studied the game coloring number of a forest:

Lemma 2 (Faigle *et al.* [3]) *Let T be a forest. Then $col_g(T) \leq 4$.*

Hence combining these two lemmas with Theorem 1, we have

Corollary 1 ([2] + [4]) *Every planar graph with girth at least 9 (resp. 7, 5) has game coloring number at most 5 (resp. 6, 8).*

In this note, we study edge-partitions of sparse graphs, in the meaning of small maximum average degree, and derive bounds on the game coloring number.

The maximum average degree of G, denoted by $Mad(G)$ is:

$$Mad(G) = \max\{2|E(H)|/|V(H)|, H \subseteqq G\}$$

Our main result is:

Theorem 2 *Let G be a simple graph.*

1. *If $Mad(G) < \frac{32}{13}$, then G admits an edge-partition into a forest and a matching.*
2. *If $Mad(G) < \frac{30}{11}$, then G admits an edge-partition into a forest and graph with maximum degree at most 2.*
3. *If $Mad(G) < \frac{32}{11}$, then G admits an edge-partition into a forest and graph with maximum degree at most 3.*
4. *If $Mad(G) < \frac{70}{23}$, then G admits an edge-partition into a forest and graph with maximum degree at most 4.*

In contrary to Theorem 1, Theorem 2 is not restricted to planar graphs. We note however that we can not infer Theorem 1 from Theorem 2, by using the usual inequality $Mad(G) \leq 2g/(g-2)$ for every planar graph G of girth at least g.

Combining with Lemmas 1 and 2, we get:

Corollary 2 *Let G be a simple graph.*

1. *If $Mad(G) < \frac{32}{13}$, then $col_g(G) \leq 5$.*
2. *If $Mad(G) < \frac{30}{11}$, then $col_g(G) \leq 6$.*
3. *If $Mad(G) < \frac{32}{11}$, then $col_g(G) \leq 7$.*
4. *If $Mad(G) < \frac{70}{23}$, then $col_g(G) \leq 8$.*

Section 2 is dedicated to the proof of Theorem 2. Section 3 contains some final remarks.

2 Proof of Theorem 2

Let G be a simple graph. Let $d(x)$ denote the degree of x in G. A vertex of degree k (resp. at least k, at most k) is called a k-*vertex* (resp. $^{\geq}k$-*vertex*, $^{\leq}k$-*vertex*). An (a, b)-*alternating cycle* is an even cycle $x_1x_2x_3 \ldots x_{2k}x_1$ such that $d(x_i) = a$ if i is even and $d(x_i) = b$ otherwise. An k_l-vertex is a vertex of degree k adjacent to exactly l 2-vertices.

Let G be a counterexample of Theorem 2, i.e. a graph that does not admit an edge-partition into a forest and a subgraph with maximum degree k ($k = 1, 2, 3, 4$), minimizing $\sigma(G) = |V(G)| + |E(G)|$.

2.1 Structural properties of G

Claim 1 *The counterexample G does not contain:*

1. 1-*vertices,*
2. *two adjacent* $^{\leq}k + 1$-*vertices,*
3. $(k + 2, 2)$-*alternating cycles.*

Proof

1. By contradiction, assume that G contains an 1-vertex v adjacent to u. By minimality, of G, the graph $H = G \setminus u$ admits an edge-partition into a forest F and a subgraph D with maximum degree k. We can extend this edge-partition to G by adding the edge uv into F, a contradiction.
2. Assume that G contains two adjacent $^{\leq}k+1$-vertices, say u an v. By minimality, of G, the graph $H = G \setminus uv$ admits an edge-partition into a forest F and a subgraph D with maximum degree k. If at least one of u and v is incident to k edges in D, then add uv in F ; otherwise, add uv into D. This extends the edge-partition to G, a contradiction.
3. Assume that G contains a $(k + 2, 2)$-alternating cycle $C = x_1x_2x_3 \ldots x_{2l}x_1$ with $d(x_i) = k + 2$ if i is even and $d(x_i) = 2$ otherwise. By minimality of G, the graph $H = G \setminus \{x_1x_2, x_2x_3, \ldots, x_{2l-1}x_{2l}, x_{2l}x_1\}$ admits an edge-partition into a forest F and a subgraph D with maximum degree k. We may assume that x_{2j} is incident to at least one edge of F, for otherwise we can add an arbitrary edge incident to x_{2j} into F. Now by adding the edges $x_{2j}x_{2j+1}$ into D, adding $x_{2j}x_{2j-1}$ into F, we obtain a required edge-partition of G, a contradiction.

$\square$

2.2 Discharging procedures

In what follows, we will define an additional structure, called *bank*, which is a subgraph of G composed of maximal connected components, called *agencies*. In fact, we will show that each bank is a forest and each agency a tree. These structures will be used, during the discharging procedure, to transfer charges. Usually, the discharging rules operate locally ; agencies will allow us to transfer charges non locally. In our discharging procedures, the vertices adjacent to an agency C will give their excess charge to C which will redistribute this excess charge to the vertices of C which does not have enough charges.

First we assign to each vertex v a charge $\omega(v)$ equal to its degree, i.e. $\forall v \in V(G), \omega(v) = d(v)$. Moreover we assign to each agency C (that will be defined later) a charge $\omega(C) = 0$. We define then discharging rules and redistribute charges accordingly. Once the discharging is finished, a new charge function ω^* is produced. However, the total sum of charges is kept fixed when the discharging is in process. Nevertheless, we can show that $\omega^*(v) \geq \frac{32}{13}$ (resp. $\frac{30}{11}$, $\frac{32}{11}$, $\frac{70}{23}$) for all $v \in V(G)$ and $\omega^*(C) \geq 0$ for all agency C of G. Hence the following equation follows:

$$\frac{32}{13}|V(G)| \leq \sum_{v\in V(G)} \omega^*(v) + \sum_{C \text{ agency of G}} \omega^*(C) = \sum_{v\in V(G)} \omega(v) + \sum_{C \text{ agency of G}} \omega(C) = \sum_{v\in V(G) d(v)} = 2|E(G)|$$

This leads to the following obvious contradiction:

$$\frac{32}{13} = \frac{\frac{32}{13}|V(G)|}{|V(G)|} \leq \frac{2|E(G)|}{|V(G)|} \leq \mathit{Mad}(G) < \frac{32}{13}$$

and hence demonstrates that no such counterexample can exist (as well for $\mathit{Mad}(G) < \frac{30}{11}, \frac{32}{11}, \frac{70}{23}$).

2.2.1 Graphs with *Mad* $< \frac{32}{13}$

Here, the *bank* of G is the subgraph of G defined as follows: its set of vertices contains all the 3_2-vertices, 3_3-vertices and the 2-vertices adjacent to 3_2-vertices, or 3_3-vertices ; its set of edges is the set of edges between the 2-vertices and the 3_2-vertices, 3_3-vertices. By Claim 1.3, an *agency* is a tree whose each leaf is a 2-vertex.

We say that a vertex, which does not belong to an agency, is *adjacent to an agency* if it is adjacent to a 2-vertex belonging to an agency.

The discharging rules are defined as follows:

R1. Every $\geq$3-vertex gives $\frac{3}{13}$ to each adjacent 2-vertex.

R2. Every $\geq$3-vertex not belonging to an agency gives $\frac{2}{13}$ to each adjacent agency.

R3. Each agency gives $\frac{2}{13}$ to each of its own 3_3-vertices.

Let us check first that for each vertex v, $\omega^*(v) \geq \frac{32}{13}$. Let v be a k-vertex ($k \geq 2$ by Claim 1.1).

Case $k = 2$ Initially, $\omega(v) = 2$. The vertex v receives $\frac{3}{13}$ from of each of its neighbors (which are $\geq$3-vertices by Claim 1.2). Hence, $\omega^*(v) = 2 + 2 \cdot \frac{3}{13} = \frac{32}{13}$.

Case $k = 3$ Initially, $\omega(v) = 3$. If v is adjacent to at most one 2-vertex, then $\omega^*(v) \geq 3 - \frac{3}{13} - \frac{2}{13} \geq \frac{34}{13}$. If v is an 3_2-vertex, then v belongs to an agency and gives two times $\frac{3}{13}$ by R1 and nothing by R2. Hence $\omega^*(v) = 3 - 2 \cdot \frac{3}{13} = \frac{33}{13}$. Finally assume that v is a 3_3-vertex. The vertex v gives three times $\frac{3}{13}$ by R1 and receives $\frac{2}{13}$ by R3. Hence $\omega^*(v) = 3 - 3 \cdot \frac{3}{13} + \frac{2}{13} = \frac{32}{13}$.

Case $k \geq 4$ Initially, $\omega(v) = k$. The vertex v is adjacent to at most k 2-vertices and to at most k agencies. Hence by R1 and R2, $\omega^*(v) \geq k - k \cdot \frac{3}{13} - k \cdot \frac{2}{13} = \frac{8k}{13} \geq \frac{32}{13}$ if $k \geq 4$.

It remains to prove that the charge remaining on each agency is non-negative. Let C be an agency. Let $n_{3_3}(C)$, and $n_l(C)$ be the number of 3_3-vertices, and leaves of C respectively. Observe that:

$$n_l(C) \geq n_{3_3}(C) \tag{1}$$

By R2, the agency C receives $\frac{2}{13}$ from its adjacent vertices (i.e. it receives $n_l(C) \cdot \frac{2}{13}$), and gives $\frac{2}{13}$ to each of its own 3_3-vertices by R3 (i.e. it gives $n_{3_3}(C) \cdot \frac{2}{13}$). Hence, $\omega^*(C) = n_l(C) \cdot \frac{2}{13} - n_{3_3}(C) \cdot \frac{2}{13} \geq n_{3_3}(C) \cdot \frac{2}{13} - n_{3_3}(C) \cdot \frac{2}{13} \geq 0$ by Equation (1). This completes the proof of Theorem 2.1.

2.2.2 Graphs with *Mad* $< \frac{30}{11}$

Here, the *bank* of G is the subgraph of G defined as follows: its set of vertices contains all the 4_3-vertices, 4_4-vertices and the 2-vertices adjacent to 4_3-vertices, or 4_4-vertices ; its set of edges is the set of edges between the 2-vertices and the 4_3-vertices, 4_4-vertices. By Claim 1.3, an *agency* is a tree whose each leaf is a 2-vertex.

The discharging rules are defined as follows:

R1. Every $^{\geq}4$-vertex gives $\frac{4}{11}$ to each adjacent 2-vertex.

R2. Every $^{\geq}4$-vertex not belonging to an agency gives $\frac{1}{11}$ to each adjacent agency.

R3. Each agency gives $\frac{2}{11}$ to each of its own 4_4-vertices.

Let us check first that for each vertex v, $\omega^*(v) \geq \frac{30}{11}$. Let v be a k-vertex.

Case $k = 2$ Initially, $\omega(v) = 2$. The vertex v receives $\frac{4}{11}$ from of each of its neighbors (which are $^{\geq}4$-vertices by Claim 1.2). Hence, $\omega^*(v) = 2 + 2 \cdot \frac{4}{11} = \frac{30}{11}$.

Case $k = 3$ Initially, $\omega(v) = 3$. The vertex v is not affected by the discharging procedure ; hence $\omega^*(v) = \omega(v) = 3 \geq \frac{30}{11}$.

Case $k = 4$ Initially, $\omega(v) = 4$. Suppose that v is a 4-vertex adjacent to at most two 2-vertices. Then v gives at most two times $\frac{4}{11}$ to its adjacent 2-vertices and two times $\frac{1}{11}$ to its adjacent agencies. Hence $\omega^*(v) \geq 4 - 2 \cdot \frac{4}{11} - 2 \cdot \frac{1}{11} = \frac{34}{11} \geq \frac{30}{11}$. Assume that v is a 4_3-vertex. Then v belongs to an agency ; hence v gives three times $\frac{4}{11}$ to its adjacent 2-vertices by R1 and nothing by R2. So $\omega^*(v) = 4 - 3 \cdot \frac{4}{11} = \frac{32}{11} \geq \frac{30}{11}$. Finally, assume that v is a 4_4-vertex. The vertex v gives four times $\frac{4}{11}$ to its incident 2-vertices by R1 and receives $\frac{2}{11}$ from its agency by R3. Hence $\omega^*(v) = 4 - 4 \cdot \frac{4}{11} + \frac{2}{11} = \frac{30}{11}$.

Case $k \geq 5$ Initially, $\omega(v) = k$. The vertex v is adjacent to at most k 2-vertices and to at most k agencies. Hence by R1 and R2, $\omega^*(v) \geq k - k \cdot \frac{4}{11} - k \cdot \frac{1}{11} = \frac{6k}{11} \geq \frac{30}{11}$ if $k \geq 5$.

It remains to prove that the charge remaining on each agency is non-negative. Let C be an agency. Let $n_{4_4}(C)$, $n_{4_3}(C)$, and $n_l(C)$ be the number of 4_4-vertices, 4_3-vertices, and leaves of C respectively.

Observe that:

$$n_l(C) \geq n_{4_3}(C) + 2 \cdot n_{4_4}(C) \tag{2}$$

By R2, the agency C receives $\frac{1}{11}$ from its adjacent vertices (i.e. it receives $n_l(C) \cdot \frac{1}{11}$), and gives $\frac{2}{11}$ to each of its own 4_4-vertices by R3 (i.e. it gives $n_{4_4}(C) \cdot \frac{2}{11}$). Hence, $\omega^*(C) = n_l(C) \cdot \frac{1}{11} - n_{4_4}(C) \cdot \frac{2}{11} \geq 2 \cdot n_{4_4}(C) \cdot \frac{1}{11} - n_{4_4}(C) \cdot \frac{2}{11} \geq 0$ by Equation (2). This completes the proof of Theorem 2.2.

2.2.3 Graphs with *Mad* $< \frac{32}{11}$

Here, the *bank* of G is the subgraph of G defined as follows: its set of vertices contains all the 5_4-vertices, 5_5-vertices and the 2-vertices adjacent to 5_4-vertices, or 5_5-vertices ; its set of edges is the set of edges between the 2-vertices and the 5_4-vertices, 5_5-vertices. By Claim 1.3, an *agency* is a tree whose each leaf is a 2-vertex.

The discharging rules are defined as follows:

R1. Every $^{\geq}5$-vertex gives $\frac{5}{11}$ to each adjacent 2-vertex.

R2. Every $^{\geq}5$-vertex not belonging to an agency gives $\frac{2}{33}$ to each adjacent agency.

R3. Each agency gives $\frac{2}{11}$ to each of its own 5_5-vertices.

Let us check first that for each vertex v, $\omega^*(v) \geq \frac{32}{11}$. Let v be a k-vertex.

Case $k = 2$ Initially, $\omega(v) = 2$. The vertex v receives $\frac{5}{11}$ from of each of its neighbors (which are $^{\geq}5$-vertices by Claim 1.2). Hence, $\omega^*(v) = 2 + 2 \cdot \frac{5}{11} = \frac{32}{11}$.

Case $k = 3, 4$ The vertex v is not affected by the discharging procedure ; hence $\omega^*(v) = \omega(v) \geq \frac{32}{11}$.

Case $k = 5$ Initially, $\omega(v) = 5$. Assume that v is a 5-vertex ajacent to at most three 2-vertices. Then by R1 and R2 $\omega^*(v) \geq 5 - 3 \cdot \frac{5}{11} - 3 \cdot \frac{2}{33} = \frac{38}{11}$. Assume now that v is a 5_4-vertex. Then v belongs to an agency and gives nothing by R2. So $\omega^*(v) = 5 - 4 \cdot \frac{5}{11} = \frac{35}{11}$. Assume finally that v is 5_5-vertex. By R1 and R3 $\omega^*(v) = 5 - 5 \cdot \frac{5}{11} + \frac{2}{11} = \frac{32}{11}$.

Case $k \geq 6$ Initially, $\omega(v) = k$. The vertex v is adjacent to at most k 2-vertices and to at most k agencies. Hence by R1 and R2, $\omega^*(v) \geq k - k \cdot \frac{5}{11} - k \cdot \frac{2}{33} = \frac{16k}{33} \geq \frac{32}{11}$ if $k \geq 6$.

It remains to prove that the charge remaining on each agency is non-negative. Let C be an agency. Let $n_{5_5}(C)$ and $n_l(C)$ be the number of 5_5-vertices and leaves of C respectively. Observe that:

$$n_l(C) \geq 3 \cdot n_{5_5}(C) \tag{3}$$

By R2, the agency C receives $\frac{2}{33}$ from its adjacent vertices (i.e. it receives $n_l(C) \cdot \frac{2}{33}$), and gives $\frac{2}{11}$ to each of its own 5_5-vertices by R3 (i.e. it gives $n_{5_5}(C) \cdot \frac{2}{11}$). Hence, $\omega^*(C) = n_l(C) \cdot \frac{2}{33} - n_{5_5}(C) \cdot \frac{2}{11} \geq 3 \cdot n_{5_5}(C) \cdot \frac{2}{33} - n_{5_5}(C) \cdot \frac{2}{11} \geq 0$ by Equation (3). This completes the proof of Theorem 2.3.

2.2.4 Graphs with *Mad* $< \frac{70}{23}$

Here, the *bank* of G is the subgraph of G defined as follows: its set of vertices contains all the 6_6-vertices and the 2-vertices adjacent to 6_6-vertices ; its set of edges is the set of edges between the 2-vertices and the 6_6-vertices. By Claim 1.3, an *agency* is a tree whose each leaf is a 2-vertex.

The discharging rules are defined as follows:

R1. Every $^{\geq}6$-vertex gives $\frac{12}{23}$ to each adjacent 2-vertex, and $\frac{1}{69}$ to each adjacent 3-vertex.

R2. Every $^{\geq}6$-vertex not belonging to an agency gives $\frac{1}{23}$ to each adjacent agency.

R3. Each agency gives $\frac{4}{23}$ to each of its own 6_6-vertices.

Let us check first that for each vertex v, $\omega^*(v) \geq \frac{70}{23}$. Let v be a k-vertex.

Case $k = 2$ Initially, $\omega(v) = 2$. The vertex v receives $\frac{12}{23}$ from of each of its neighbors (which are $^{\geq}6$-vertices by Claim 1.2). Hence, $\omega^*(v) = 2 + 2 \cdot \frac{12}{23} = \frac{70}{23}$.

Case $k = 3$ The vertex v is adjacent to $^{\geq}6$-vertices by Claim 1.2. Then v receives $\frac{1}{69}$ from each of its neighbors ; hence $\omega^*(v) = 3 + 3 \cdot \frac{1}{69} = \frac{70}{23}$.

Case $k = 4, 5$ The vertex v is not affected by the discharging procedure ; hence $\omega^*(v) = \omega(v) \geq \frac{70}{23}$.

Case $k = 6$ Initially, $\omega(v) = 6$. Assume that v is adjacent to at most five 2-vertices. Then by R1 an R2 $\omega^*(v) \geq 6 - 5 \cdot \frac{12}{23} - 5 \cdot \frac{1}{23} = \frac{73}{23}$. Assume finally that v is a 6_6-vertex. The vertex v belongs to an agency. By R1 and R3 $\omega^*(v) = 6 - 6 \cdot \frac{12}{23} + \frac{4}{23} = \frac{70}{23}$.

Case $k \geq 7$ Initially, $\omega(v) = k$. The vertex v is adjacent to at most k $^{\leq}3$-vertices and to at most k agencies. Hence by R1 and R2, $\omega^*(v) \geq k - k \cdot \frac{12}{23} - k \cdot \frac{1}{23} = \frac{10k}{23} \geq \frac{70}{23}$ if $k \geq 7$.

It remains to prove that the charge remaining on each agency is non-negative. Let C be an agency. Let $n_{6_6}(C)$ and $n_l(C)$ be the number of 6_6-vertices and leaves of C respectively.

Observe that:

$$n_l(C) \geq 4 \cdot n_{6_6}(C) \tag{4}$$

By R2, the agency C receives $\frac{1}{23}$ from its adjacent vertices (i.e. it receives $n_l(C) \cdot \frac{1}{23}$), and gives $\frac{4}{23}$ to each of its own 6_6-vertices by R3 (i.e. it gives $n_{6_6}(C) \cdot \frac{4}{23}$). Hence, $\omega^*(C) = n_l(C) \cdot \frac{1}{23} - n_{6_6}(C) \cdot \frac{4}{23} \geq 4 \cdot n_{6_6}(C) \cdot \frac{1}{23} - n_{6_6}(C) \cdot \frac{4}{23} \geq 0$ by Equation (4). This completes the proof of Theorem 2.4.

3 Conclusion

In this note, we established that for every simple graph G, if $\mathit{Mad}(G) \leq \frac{32}{13}$ (resp. $\frac{30}{11}$, $\frac{32}{11}$, $\frac{70}{23}$) then G admits an edge-partition into a forest and a graph with maximum degree at most 1 (resp. 2, 3, 4), hence the game chromatic of G is at most 5 (resp. 6, 7, 8). In order to study the tightness of Theorem 2, we introduce a function $f : \mathbb{N} \to \mathbb{R}$ defined by $f(k) = \inf\{\mathit{Mad}(H) \mid H$ does not admit an edge-partition into a forest and a subgraph with maximum degree k $\}$. It is easy to observe that the complete bipartite graph $K_{2,2k+2}$ has $\mathit{Mad}(K_{2,2k+2}) = \frac{4(k+1)}{k+2}$ and does not admit an edge-partition into a forest and a subgraph with maximum degree at most k. Hence,

$$\begin{aligned}
2,461... = \frac{32}{13} &\leq f(1) \leq \frac{8}{3} = 2,666...\\
2,727... = \frac{30}{11} &\leq f(2) \leq 3\\
2,909... = \frac{32}{11} &\leq f(3) \leq \frac{16}{5} = 3,2\\
3,043... = \frac{70}{23} &\leq f(4) \leq \frac{10}{3} = 3,333...
\end{aligned}$$

We conclude with the following problem:

Problem 1 *For every k, what are (if any) the graphs which do not admit an edge-partition into a forest and a subgraph with maximum degree at most k, but such that every graph with smaller maximal average degree does?*

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